A Novel Method For Designing Compressed Sensing Matrix

Shota Ishikawa
Faculty of Systems Engineering
Wakayama University
Wakayama, Japan
s151001@center.wakayama-u.ac.jp

Waiyuan Wu
Faculty of Systems Engineering
Wakayama University
Wakayama, Japan
wuhy@center.wakayama-u.ac.jp

Yankun Lang
Faculty of Systems Engineering
Wakayama University
Wakayama, Japan
joechrislyk@yahoo.co.jp

***Abstract*—Compressed Sensing (CS) is a method to restore sparse signals from a small number of observed components using a CS-matrix, and is usually applied to data compression. In previous studies, Random-Matrix, whose elements are randomly generated variables, is considered to be the most suitable form for CS-Matrix. However, it cannot ensure the availability of CS due to its randomness. Moreover, it has a problem with the reproducibility. To solve this problem, in this paper, we propose a novel generation method of CS-Matrix with randomness removed. Through numerous experiments with numerical value and images, we confirmed the reconstruction precision using proposal CS-matrix were higher than Random-Matrix.**

***Index Terms*—Compressed Sensing; CS matrix; Low incoherence Random-Matrix; Image**

I. Introduction

In this paper, we propose a method to compress images by using compressed sensing. Compressed sensing is a signal processing technique for reconstructing a high dimensional signal from fewer samples than required by the Shannon-Nyquist sampling theory. For a given n dimension vector $\boldsymbol{x}$ with a matrix $\boldsymbol{A}$(with n columns and m rows), where $m \ll n$, it can be compressed as a result of a low dimension vector $\boldsymbol{y}$ m dimension by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{A}$ is called compressed sensing matrix (CS- matrix). Generally, this linear underdetermined system has an infinite number of solutions to $\boldsymbol{x}$. However, according to compressed sensing theory, if $\boldsymbol{x}$ is a sparse signal and $\boldsymbol{A}$ is an incoherence matrix, $\boldsymbol{x}$ can be recovered.

The first condition can be achieved by transforming $\boldsymbol{x}$ onto a sparse domain, which is usually done by using DCT (Discrete Cosine Transform) or DWT (Discrete Wavelet Transform). For the second condition, in previous studies [1] [2] [4], Random-Matrix, in which each column is a randomly generated vector, is considered to be the most appropriate form of CS-Matrix since the coherence between its columns is very low. However, it cannot ensure the availability of CS due to its randomness. [2] has proved that a Random Matrix is available only under the condition of

$$m \geq Cslog(n/s), \tag{2}$$

where s is the sparsity of $\boldsymbol{x}$, meaning the number of non-zeros in it. C is a constant depending on each instance, and its value needs to be determined from a large number of experiments, which increases the complexity.Moreover, reconstruction of the signal is very sensitive to each column of Random Matrix. Any inappropriate random vector can lead to a failure result. To solve these problems, in this paper, we propose a novel generation method of incoherence CS-Matrix without randomness. We use OMP (Orthogonal Matching Pursuit) method [5] for signal reconstruction.

II. Our CS-matrix

In this section, we produce some new CS-matrices instead of random matrix. Our proposal matrices have low incoherence.To make sure of the CS-matrixes have low in-coherence. An identity matrix can be considered as a part of the CS-matrix since the coherence of it is 0. At the same time, we append some vectors to it under a low coherence.

A. CS-matrix and identity matrix

Since the coherence of an identity matrix is 0, it can be considered as a part of a CS-matrix. However, the identity matrix has no compressibility even though a sparse signal can be correctly reconstructed with it. This problem can be simply solved by appending some vectors to the identity matrix under a low coherence. Our proposal CS-matrix is given by the following form:

$$\boldsymbol{A} = [\boldsymbol{E}|\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_{n-m}], \tag{3}$$

where $\boldsymbol{E}$ is a $m \times m$ identity matrix. $\boldsymbol{v}_i$ is the m-dimensional appended vector with low coherence. The dimension of $\boldsymbol{A}$ is $m{\times}n$, where $m \ll n$. For better understanding, in this paper, we only discuss the case of two orthogonal vectors, which $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are used.

B. Details of our proposal CS-matrix

In our proposal CS-matrix, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ can designed as orthogonal vectors so that they will be independent. To make the coherence between vectors and each column of the identity matrix [$\boldsymbol{E}$ low, each non-zero element in them is no more than 1. However, these conditions cannot guarantee the validity of

978-1-5386-2615-3/18/$31.00 2018 IEEE

our proposal CS-Matrix yet. We give a simple example to show the problem. Consider a CS-Matrix $\boldsymbol{P}$ given by:

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{4}$$

which is constituted by a 6*6 identity matrix and two 6-dimensional orthogonal vectors where each element is 1. Each of them and each column of $\boldsymbol{E}$ has low incoherence. They divides vectors into odd and even so that this matrix can compress vectors. It has been observed from a large number of experiments that some specific vectors $\boldsymbol{x}$ cannot be recovered with this matrix. We list the common characteristic among them as follows:

I. Both indices of the two nonzero elements are either odd or even.

Ⅱ.Signs of the two non-zero elements are same.

Vector holding these characteristics at the same time cannot be recovered with CS-matrix $\boldsymbol{P}$. We give a simple example to explain the reason. For a signal $\boldsymbol{x} = [2\ 0\ 3\ 0\ 0\ 0\ 0\ 0]^T$, it will be compressed as $\boldsymbol{y} = [2\ 0\ 3\ 0\ 0\ 0]^T$. In the reconstruction procedure, according to OMP algorithm, $\hat{\boldsymbol{x}}$ is supposed to be recovered from the first and third columns of $\boldsymbol{A}$, whose product is 2 and 3 respectively. However, the largest product comes from the seventh column, which will cause the reconstruction failed. In section 3, we list more failure examples caused by the same reason.

To solve this problem, $\boldsymbol{A}$ is modified to be the following form:

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.5 \end{pmatrix}. \tag{5}$$

For the same signal $\boldsymbol{x} = [2\ 0\ 3\ 0\ 0\ 0\ 0\ 0]^T$, after the compression, the product of the seventh column of $\boldsymbol{Q}$ with $\boldsymbol{y}$ becomes 2.5, which is not the biggest, thus this column won't be selected and $\boldsymbol{x}$ can be correctly recovered. As shown in section 3, this CS-Matrix can be used for compressing image and fluid simulation data. Although some failed results were obtain when using this matrix, but $\boldsymbol{Q}$ shows a high success rate in all combination of non-zero elements of $\boldsymbol{x}$.

TABLE I: RESULT BY USING RANDOM MATRIX $\boldsymbol{R}$

First element	Second element							
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	2000	2000	1884	1898	1812	1866	1920	2000
2		2000	1755	2000	1725	1883	1841	2000
3			2000	2000	1757	1966	2000	1999
4				2000	2000	1952	1970	2000
5					2000	1880	1613	1918
6						2000	2000	2000
7							2000	2000
8								2000

TABLE II: RESULT BY USING OUR MATRIX $\boldsymbol{P}$

First element	Second element							
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	2000	2000	524	1999	1015	2000	2000	2000
2		2000	2000	513	1999	993	2000	2000
3			2000	2000	1152	2000	2000	2000
4				2000	2000	1136	2000	2000
5					2000	2000	2000	2000
6						2000	2000	2000
7							2000	2000
8								2000

TABLE III: RESULT BY USING OUR MATRIX $\boldsymbol{Q}$

First element	Second element							
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	2000	2000	1999	2000	2000	2000	1808	2000
2		2000	2000	1999	2000	2000	2000	1790
3			2000	2000	2000	2000	1775	2000
4				2000	2000	1999	2000	1784
5					2000	2000	1796	2000
6						2000	2000	1807
7							2000	2000
8								2000

III. Experiments

In this section, first, we used several random matrices to compress a large group of signals (each in the form of vectors) and show the recovery results. Then, the best one $\boldsymbol{R}$ is selected to compare with the two proposal CS-Matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$. At last, evaluation experiments are implemented by using the two matrices to compress and reconstruct image data.

We compressed several 8-dimensional signals with 20 random matrices as well as the proposal CS-matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$, and evaluate the reconstruction performance respectively. Each signal $\boldsymbol{x}$ is 8-dimension and 2-sparse, with the elements bounded between [-500, 499]. The locations of the two non-zero elements have 36 combinations. For each specific combination, 2000 corresponding signal were generated and then compressed. The signal will be viewed as successfully recovered only if the Euclid error of each non-zero element was lower than 0.1.

Table I 寶 Table Ⅱ show the number of the signals successfully recovered by using best random $\boldsymbol{R}$, our CS-matrix $\boldsymbol{P}$, and $\boldsymbol{Q}$ respectively. In each table, the vertical axis shows the index of the first non-zero element, and the horizontal axis shows that of the second. The recovery rate is calculated by:

$$(recovery\ rate) = s/a * 100(\%), \tag{6}$$

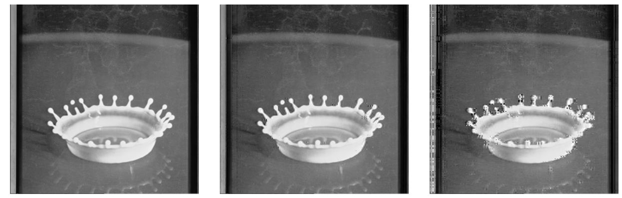
where s is the number of signals Successful recovery in all combination, and a is the product the number of all signals used in this experiment.

(a) original (b) our $\boldsymbol{Q}$ (c) random $\boldsymbol{R}$

Fig. 1: Experimental result of LENNA

(a) original (b) our $\boldsymbol{Q}$ (c) random $\boldsymbol{R}$

Fig. 2: Experimental result of AIRPLANE



(a) original (b) our $\boldsymbol{Q}$ (c) random $\boldsymbol{R}$

Fig. 3: Experimental result of SPLASH

(a) original (b) our $\boldsymbol{Q}$ (c) random $\boldsymbol{R}$

Fig. 4: Enlargement Figures of result of LENNA (Fig.1)

Table I shows the result obtained by using the random matrix $\boldsymbol{R}$. Only 69639 signals were successfully recovered. As we discussed in Section 1, the randomness, which is an intrinsic property of all Random matrices, is the primary reason for causing a low recovery rate (96.72%).

The result obtained by using CS-matrix $\boldsymbol{P}$ is shown in table 2, where it can be clearly observed that signals with characteristic I as described in subsection 2.2 had a low probability to be recovered. The reason of the failure was discussed in subsection 2.2. For the signals whose non-zero elements have a high index (7 or 8) , they were successfully recovered regardless of those two characteristics. The number of success was 65331, and the recovery rate is 90.74%, which is lower than that by using $\boldsymbol{R}$. Meanwhile CS-matrix $\boldsymbol{P}$ has a lot of limitations therefore it is not applicable.

At last, we show the result of $\boldsymbol{Q}$ in table 3, the higher value was illustrated by red color and the smaller one was blue by comparing with the result of $\boldsymbol{R}$. Apparently, more signals(70757 signals) were successfully recovered and the recovery rates achieved 97.78%. Additionally, another advantage of this matrix is that it can recover any sort of 2-sparse signals without considering the locations of its non-zeros elements.

A. Compressing image datas

At last, we show the result of image data compression with two matrixes. The dataset we used is ”The USC-SIPI Image Database”[13]. Before the compression, all images were transform onto a sparse domain by using Discrete Cosine Transform (DCT), then shrunk. The recovery results are shown in Fig.1-3, from which it can be clearly observed that the proposal CS-Matrix gained a better performance. Fig.4 is the Enlargement Figures of result of LENNA. As you can see, result of our matrix has less error than random one.

Additionally, we compressed image data with 12*16 matrix $\boldsymbol{Q}^{'}$ which bases on same idea with $\boldsymbol{Q}$. We compressed same image data with same procedure. To compare with result of $\boldsymbol{Q}^{'}$ we made 12*16 random matrix $\boldsymbol{R}^{'}$. The results are shown in Fig.5-7.

IV. Conclusion

In this paper, we have proposed a novel CS-Matrix with randomness removed. It has been proved that our matrix is able to compress and reconstruct signals in the form of vectors and images. Meanwhile, through numerous experiments, our matrix has achieved a higher recovery accuracy by comparing with traditional random matrix. However, there are still two problems need to be solved.

The first problem is that our matrix still cause failure in some cases. The second problem is about the compression ratio, which is only 25%. Our further work will focus on solving these problems by adding the number of the columns of $\boldsymbol{A}^{'}$ as well as improving the recovery method.

Acknowledgment

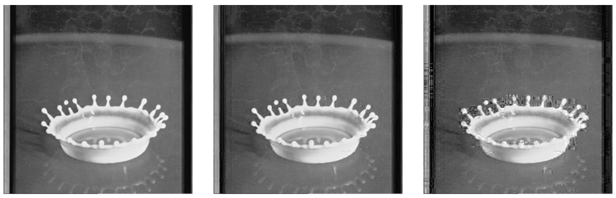
This work is partially supported by JSPS KAKENHI Grant #26330195 to Q. Chen and JSPS KAKENHI Grant #15K01331 to H. Wu.

(a) original (b) our $\boldsymbol{Q}$ (c) random $\boldsymbol{R}$

Fig. 5: Experimental result of LENNA with larger matrices

(a) original (b) our ***Q*** (c) random ***R***

Fig. 6: Experimental result of AIRPLANE with larger matrices



(a) original (b) our ***Q*** (c) random ***R***

Fig. 7: Experimental result of SPLASH with larger matrices

REFERENCES

[1] E.Candes, Justin Romberg, and T.Tao, 褰彝 bust uncertainly principles: Exact signal reconstruction from hghly incomplete frequency information 褰, IEEE Trans InformThory, vol.52 no.2, pp.489-509, Feb. 2006.

[2] E.J.Cand 褰册 s, M.B. Wakin, 褰戡 n Introduction to compressive sampling 褰, IEEE Signal Processing Magazine, Vol.25, No.2, pp.21-30, March 2008

[3] D. Donoho, 褰廚 ompressed sensing, 褰 IEEE Trans. Inform. Theory, vol. 52, no. 4, pp. 1289-1306, Apr. 2006.

[4] Akira HIRABAYASHI ☒娱弾 Tutorial] Compressed sensing 褰 Basic principle and state-of-the art results 褰哲栗 EICE ☒靴 AS2009-☒祁 LD2009-1 ☒ (2009) ☒

[5] Kazushi MIMURA ☒娱戮 ompressed sensing 褰 Sparse recovery and its algorithms 褰凧哲訓 ublications of the research institute for mathematical sciences, No.1803, pp.26-56, (2012.08)

[6] M. Bayati and A. Montanari, 褰弋 he dynamics of message passing on dense graphs, with applications to compressed sensing, 褰 Proc. of the 2010 IEEE Int 褰冤 Sympo. on Info. Theory (ISIT), 1528 (2010). [the longer version will appear inIEEE Transactions on Information Theory]

[7] Yoshiyuki Kabashima, 褰彝 tatistical-mechanical approach to compressed sensing 褰, The Brain & Neural Networks, Vol. 17 (2010) No.2, pp.70-78.

[8] Takahiro YAMAMOTO and Masaki KAWAMURA ☒娱廚 onsideration of introduction of IST method to wet paper code and its problem 褰 IEICE technical report. EMM, Enriched multimedia, vol.113, no.66, pp.19-24 ☒ (2013-05-17)

[9] Kenichi IWATA, Masakatsu MORII, Tomohiko UEMATSU, 褰 Putting files on a diet -Fundamentals of file compression-褰哲卦 HE JOURNAL OF INFORMATION SCIENCE AND TECHNOLOGY ASSOCIATION ☒祁 ol.47, no.1 ☒継 p.17-23 ☒ (1997) ☒

[10] Masaaki KOBAYASHI, Seiichiro KAMATA, 褰廰 ossless compression method for color document images 褰, ICICE, Vol.J85-DII, No.4, pp.584-597, (2002) ☒

[11] Mitsunori KORI, 褰戡 High-performance data compresstion for data warehouse 褰, ICICE, Vol47, No.SIG 13(TOD 31), (2006) ☒

[12] Yoshio SUGIMORI, Yoshihide KIMATA, Hiroya ARAKI, 褰 Compressed transmission technology of pictorial and music information in teletext 褰, The Journal of the Institute of Television Engineers of Japan, 36(8), pp.696-703, (1982-08-20)

[13] University of Southern California, ”The USC-SIPI Image Database”, http://sipi.usc.edu/database/(accessed 2017-06-20)